



# Matrix Inverse

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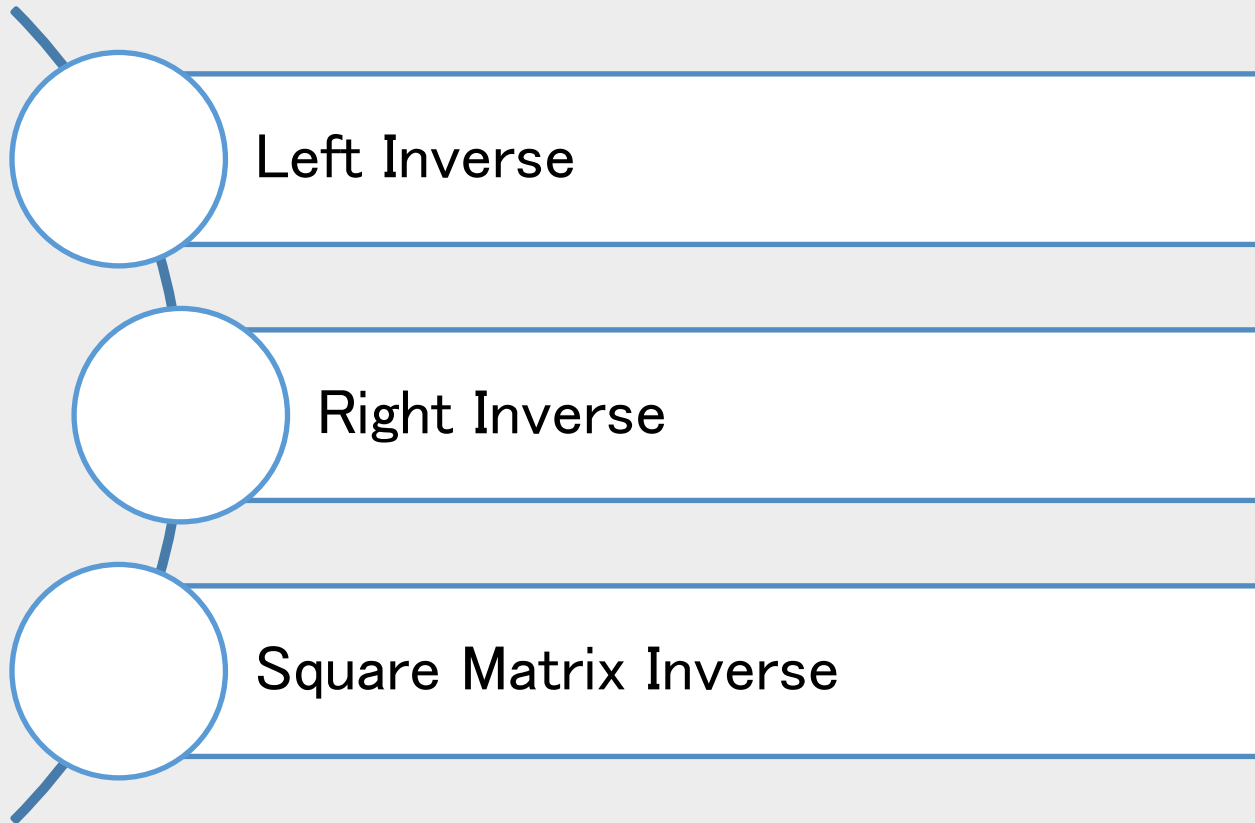
## Linear Algebra

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# Left Inverse

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## Definition

- A number  $x$  that satisfies  $xa = 1$  is called the inverse of  $a$
- Inverse (i.e.,  $\frac{1}{a}$ ) exists if and only if  $a \neq 0$ , and is unique
- A matrix  $X$  that satisfies  $XA = I$  is called a left inverse of  $A$
- If a left inverse exists we say that  $A$  is left-invertible
- $A: m \times n \Rightarrow I: n \times n \Rightarrow X: n \times m$

## Example

The matrix  $A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$

Has two different left inverses:

$$B = \frac{1}{9} \begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix},$$

$$C = \frac{1}{2} \begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$$



## Method

- ❑ Suppose  $Ax = b$ , and  $A$  has a left inverse  $C$
- ❑ Then  $Cb = C(Ax) = (CA)x = Ix = x$
- ❑ So multiplying the right-hand side by a left inverse yields the solution



## Note

- ❑ A non-zero column vector always has a left inverse.
- ❑ Left inverse is not unique.

## Example

Type equation here.

- ❑  $a = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$  Three ways: (1)  $a^{-1} = \frac{1}{a_i} e_i^T$  where  $a_i \neq 0$  (2)  $a^T a = 1 \Rightarrow \frac{a^T}{\|a\|^2}$
- ❑ Matrix with orthonormal columns  $A^{-1} = A^T$

## Example

- ❑ Row vector does not have left inverse

$$A = [1 \quad 0 \quad 3]$$

Think about  $\text{rank}(BA)$ ,  $\text{rank}(I)$  with this theory:  $\text{rank}(BA) \leq \min(\text{rank}(A), \text{rank}(B))$



## Theorem

A matrix is left-invertible if and only if its columns are linearly independent

Proof



## Theorem

- ❑ If  $A$  has a left inverse  $C$  then the columns of  $A$  are linearly independent
- ❑ We'll see later that the converse is also true, so:

**A matrix is left-invertible if and only if its columns are linearly independent**

- ❑ Matrix generalization of

A number is invertible if and only if it is nonzero

**From Previous Theorem**

Left-invertible matrices are all tall or square

- ❑ Wide matrix is not always left invertible
- ❑ Tall or square matrices can be left invertible

## Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -1 \\ 1 & 3 & 4 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$



# Right Inverse

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## Theorem

A matrix is right-invertible if and only if its rows are linearly independent

Proof



## Definition

- A matrix  $X$  that satisfies  $AX = I$  is a right inverse of  $A$
- If a right inverse exists we say that  $A$  is right-invertible
- $A$  is right-invertible if and only if  $A^T$  is left-invertible:

$$AX = I \Rightarrow (AX)^T = I \Rightarrow X^T A^T = I$$

- so we conclude:

**$A$  is right invertible if and only if its rows are linearly independent**

- Right-invertible matrices are wide or square



## Method

- ❑ Suppose  $A$  has a right inverse  $B$
- ❑ Consider the (square or underdetermined) equations of  $Ax = b$
- ❑  $x = Bb$  is a solution:

$$Ax = A(Bb) = (AB)b = Ib = b$$

- ❑ So  $Ax = b$  has a solution for any  $b$

## Example

- ❑ Same  $A, B, C$  in last example.
- ❑  $C^T$  and  $B^T$  are both right inverses of  $A^T$
- ❑ Under-determined equations  $A^T x = (1, 2)$  has (different) solutions.

$$B^T(1, 2) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right), \quad C^T(1, 2) = \left(0, \frac{1}{2}, -1\right)$$

there are many other solutions as well

# Conclusion: Left and Right Inverse

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## Definition

**Left-Invertible matrix:** if  $X$  is a left inverse of  $A$ , then

$$Ax = b \Rightarrow x = XAx = Xb$$

There is at most one solution using  $X$  (if there is a solution, it must be equal to  $Xb$ )

We must know in advance that there exists at least one solution

Why “at most”??

$$XA = I$$

$$\begin{cases} -y_1 + y_2 = -4 \\ 0y_1 - y_2 = 3 \\ 2y_1 + y_2 = 0 \end{cases}$$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} -1 & 1 & -4 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array} \right]$$



## Note

□ If the system of equations  $Ax = b$  is consistent, and if a matrix  $B$  exists such that  $BA = I$ , then the system of equations has a unique solution, namely  $x = Bb$ .

□ **Right-inversible matrix:** if  $X$  is a right inverse of  $A$ , then there is at least one solution ( $x = Xb$ ):

$$x = Xb \Rightarrow Ax = AXb = b$$

□ To pursue these ideas further, suppose that again we want to solve a system of linear equations,  $Ax = b$ . Assume now that we have another matrix,  $B$ , such that  $AB = I$ . Then we can write  $A(Bb) = (AB)b = Ib = b$ ; hence  $Bb$  solves the equations  $Ax = b$ . This conclusion did not require an a priori assumption that a solution exist; we have produced a solution. The argument does not reveal whether  $Bb$  is the only solution. There may be others.

□ **Invertible matrix:** if  $A$  is invertible, then

$$Ax = b \Leftrightarrow x = A^{-1}b$$

There is a unique solution



- System of linear equations  $Ax = b$ :
  - A right inverse of  $A$ , say  $AB = I$ . Then  $Bb$  is a solution, as is verified by nothing  $A(Bb) = (AB)b = Ib = b$ .
  - Why don't need to check the consistency for using right inverse?
  - A left inverse of  $A$ , say  $CA = I$ , then we can only conclude that  $Cb$  is the sole candidate for a solution; however, it must be checked by substitution to determine whether, in fact, it is a solution



# Square Matrix Inverse

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## Definition

For  $A \in M_{n \times n}$ , if there exists a matrix  $B \in M_{n \times n}$  such that  $AB = BA = I_n$ , then:

- A is invertible (or nonsingular)
- B is the inverse of A
- The inverse of A is denoted by  $B = A^{-1}$

A square matrix that does not have an inverse is called non-invertible (or singular)

For a square matrix left and right inverse are the same. Rows and columns are linear independent.

## Theorem

For a square matrix, the right and left inverse are the same

## Theorem

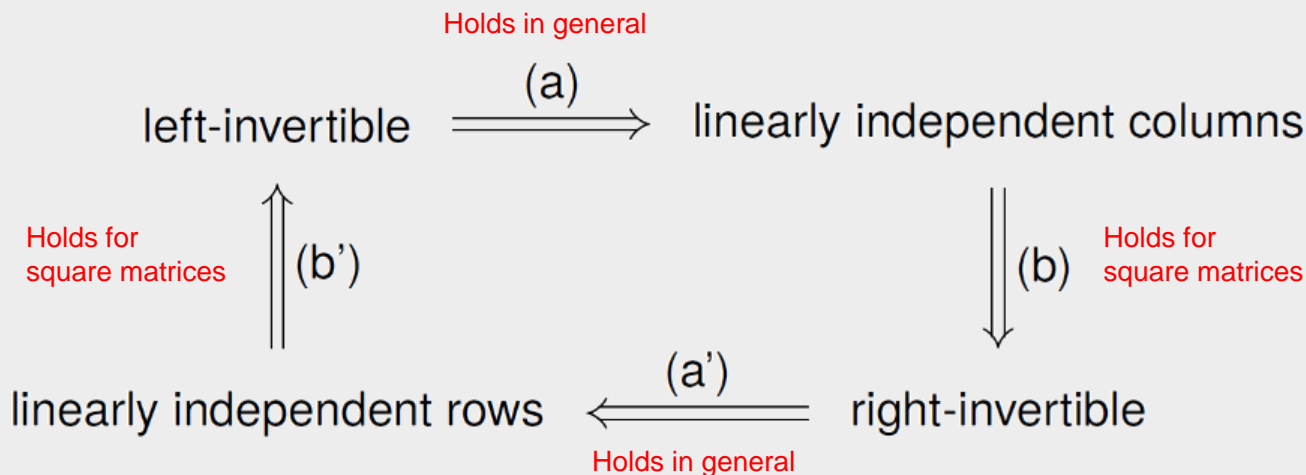
The inverse of a square matrix is unique.



## Theorem

**A square matrix is invertible if and only if its columns are linearly independent**

Proof





## Method

- ❑ Let  $A$  be a  $n \times n$  matrix:
  - ❑ Adjoin the identity  $n \times n$  matrix  $I_n$  to  $A$  to form the matrix  $[A : I_n]$ .
  - ❑ Compute the reduced echelon form of  $[A : I_n]$ .
- ❑ If the reduced echelon form is of the type  $[I_n : B]$ , then  $B$  is the inverse of  $A$ .
- ❑ If the reduced echelon form is not the type  $[I_n : B]$ , in that the first  $n \times n$  submatrix is not  $I_n$  then  $A$  has no inverse.

$$[A \mid I] \text{ Gauss–Jordan elimination } [I \mid A^{-1}]$$

## Important

An  $n \times n$  matrix is invertible if and only if its reduced echelon form is  $I_n$ .

$A$  is row equivalent to  $I_n$



## Example

Find inverse of the following matrix using Gauss-Jordan Elimination:

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

$$AX = I \Rightarrow \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equating corresponding entries we have:

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \quad (1)$$
$$\begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases} \quad (2)$$

This two system of linear equations  
have the same coefficient matrix,  
which is exactly the matrix  $A$





## Rest of The Example

Using Gauss-Jordan Elimination on the matrix  $A$  with the same row operations

$$\begin{aligned}
 (1) &\Rightarrow \left[ \begin{array}{cc|c} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array} \right] \Rightarrow \left[ \begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array} \right] \Rightarrow x_{11} = -3, x_{21} = 1 \\
 (2) &\Rightarrow \left[ \begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array} \right] \Rightarrow \left[ \begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array} \right] \Rightarrow x_{12} = -4, x_{22} = 1
 \end{aligned}$$

Thus  $X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$

Solution for  $\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$       Solution for  $\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$

$$\left[ \begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array} \right] \xrightarrow{\text{Gauss-Jordan elimination}} \left[ \begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array} \right]$$

$A$                        $I$                        $I$                        $A^{-1}$



## Definition

Properties (If  $A$  is invertible matrix,  $k$  is a positive integer and  $c$  is a scalar):

- $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$
- $A^k$  is invertible and  $(A^k)^{-1} = A^{-k} = (A^{-1})^k$
- $cA$  is invertible if  $c \neq 0$  and  $(cA)^{-1} = \frac{1}{c}A^{-1}$
- $A^T$  is invertible and  $(A^T)^{-1} = (A^{-1})^T$

## Theorem

If  $A$  and  $B$  are invertible matrices of order  $n$ , then  $AB$  is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$

$$(A_1 A_2 A_3 \cdots A_n)^{-1} = A_n^{-1} \cdots A_3^{-1} A_2^{-1} A_1^{-1}$$





## Theorem

The solution set  $K$  of any system  $Ax=b$  of  $m$  linear in  $n$  unknowns is,  $s$  is a particular solution:

$$K = s + \text{Null}(T_A)$$

## Theorem (Using above Theorem)

Let  $Ax = b$  be a system of  $n$  linear equations in  $n$  variable.

The system has exactly one solution  $A^{-1}b$  if and only if  $A$  is invertible.



## Definition

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If  $ad - bc \neq 0$ , then  $A$  is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If  $ad - bc = 0$ , then  $A$  is not invertible

## Note

Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .  $\det A = ad - bc$ .

$2 \times 2$  matrix  $A$  is invertible if and only if  $\det A \neq 0$ .



## Definition

Each Elementary Matrix  $E$  is invertible. The inverse of  $E$  is the elementary matrix of the same type that transforms  $E$  back into  $I$ .

## Example

Find the inverse of  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$



## Method

- ❑ Suppose  $A$  is invertible
- ❑ For any  $b$ ,  $Ax = b$  has the unique solution
$$x = A^{-1}b$$
- ❑ Matrix generalization of simple scalar equation  $ax = b$  having solution  $x = \left(\frac{1}{a}\right)b$  (for  $a \neq 0$ )
- ❑ Simple-looking formula  $x = A^{-1}b$  is basis for many applications



## Conclusion

The following are equivalent for a square matrix  $A$ :

- ☐  $A$  is invertible
- ☐ Columns of  $A$  are linearly independent
- ☐ Rows of  $A$  are linearly independent
- ☐  $A$  has a left inverse
- ☐  $A$  has a right inverse

$$\text{row rank}(A) = \text{col rank}(A) = n$$

If any of these hold, all others do



## Examples

□  $I^{-1} = I$

□ If  $Q$  is orthogonal, i.e. , square with  $Q^T Q = I$ , then  $Q^{-1} = Q^T$

□  $2 \times 2$  matrix  $A$  is invertible if and only if  $A_{11}A_{22} \neq A_{12}A_{21}$

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

- You need to know this formula
- There are similar but much more complicated formulas for larger matrices (and no, you do not need to know them)

□ Consider matrix  $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$

➤  $A$  is invertible, with inverse:

$$A^{-1} = \frac{1}{30} \begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}$$

- Verified by checking  $AA^{-1} = I$  (or  $A^{-1}A = I$ )
- We'll soon see how to compute the inverse



## Properties

- $(AB)^{-1} = B^{-1}A^{-1}$
- If  $A$  is nonsingular, then  $A^T$  is nonsingular  
 $(A^T)^{-1} = (A^{-1})^T$  (sometimes denoted  $A^{-T}$ )
- **Negative matrix powers:**  $(A^{-1})^k$  is denoted by  $A^{-k}$
- With  $A^0 = I$ , Identity  $A^k A^l = A^{k+l}$  holds for any integers  $k, l$



## Theorem

Lower Triangular  $L$  with non-zero diagonal entries is invertible

Proof??

## Theorem

Upper Triangular  $R$  with non-zero diagonal entries is invertible

Proof??





## Why Matrix of Change of Basis is invertible?

*Because the column and rows of it is the basis so they are linear independent and invertible*



## Theorem

Given a square matrix  $M$  and its inverse  $M^{-1}$ , then  $M$  and  $M^{-1}$  have the same rank.



## Theorem

If  $A$  is  $m \times n$  and  $B$  is an  $n \times n$  invertible matrix, then  $\text{rank}(AB) = \text{rank}(A)$ .

Solution:

$$\text{rank}(AB) \leq \text{rank}(A), [*] \text{rank}(AB) \leq \text{rank}(B)$$

$$B^{-1}(BA) = A \Rightarrow \text{rank}(B^{-1}(BA)) = \text{rank}(A) \leq \text{rank}(B^{-1})$$

$$A^{-1}(AB) = B \Rightarrow \text{rank}(A^{-1}(AB)) = \text{rank}(B) \leq \text{rank}(AB)$$

Therefore:  $\text{rank}(A) \leq \text{rank}(AB)$  [\*\*]

so using [\*,]\*\* then  $\text{rank}(A) = \text{rank}(AB)$



Let  $A$  be a square  $n \times n$  matrix. Then the following are equivalent.

- ❑ 1.  $A$  is an invertible matrix.
- ❑ 2.  $A$  is row equivalent to the  $n \times n$  identity matrix.
- ❑ 3.  $A$  has  $n$  pivot positions.
- ❑ 4. The equation  $Ax = 0$  has only the trivial solution.
- ❑ 5. The columns of  $A$  form a linearly independent set.
- ❑ 6. The linear transformation  $x \rightarrow Ax$  is one-to-one.
- ❑ 7. The equation  $Ax = b$  has at least one solution for each  $b \in \mathbb{R}^n$ .
- ❑ 8. The columns of  $A$  span  $\mathbb{R}^n$ .
- ❑ 9. The linear transformation  $x \rightarrow Ax$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .
- ❑ 10. There is an  $n \times n$  matrix  $C$  such that  $CA = I$ .
- ❑ 11. There is an  $n \times n$  matrix  $D$  such that  $AD = I$ .
- ❑ 12.  $A^T$  is an invertible matrix.